A note on p-adic uniformization

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A note on p-adic uniformization

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INTRODUCTION

Some algebraic varieties X over a non-archimedean field K admit an interesting analytic uniformization $Y \xrightarrow{\pi} X$. This means that X is seen as a rigid analytic space over K, that Y is also a rigid analytic space and that the morphism π has the property: there exists an admissible affinoid covering $\{X_i | i \in I\}$ of X such that for each $i \in I$, $\pi^{-1}(X_i)$ is isomorphic to the disjoint union of copies of X_i.

Well known examples of uniformization are: the Tate-curve, Mumford curves [4, 6], some Abelian varieties [2, 7]. A less known example is Mumford's construction of surfaces of general type [8].

In all cases it is believed that the given uniformization is the universal one. The remaining problem, which is solved in this paper, is to verify that the given Y is simply connected. For one-dimensional spaces Y, the proof is given in [4], [10].

The Grothendieck topology on higher-dimensional spaces is rather complicated. In [9] it is shown that many higher-dimensional spaces Y have trivial cohomology for constant sheaves (and other sheaves). The method used there is rich enough to show that the same spaces are simply connected. The definition of simply connected, used here, is the following:

"A connected analytic space X over K is called simply connected if the only connected analytic covering of X is equal to $X \xrightarrow{id} X$."

This notion can be translated as follows: Every sheaf of sets S on X, which is locally constant (for the Grothendieck-topology) is a constant sheaf.

The locally constant sheaf S is given by an admissible connected covering $\{X_i | i \in I\}$ and constant sheaves $S|X_i$. Then S is constant if and only if all the restriction maps $S(X) \to S(X_i)$ are bijective.

We will often use the following obvious property: Let an admissible covering $\{X_n | n \in \mathbb{N}\}$ of X be given such that

(i) every X_n is simply connected.

(ii) $X_n \subset X_{n+1}$ for all n.

Then X is simply connected.

2. MAIN RESULTS

In the sequel we will use the terminology and notations of [9].

THEOREM 1. Let $\varphi : X \to Y$ be a morphism of quasi-compact K-analytic spaces such that:

(i) Y is simply connected.

(ii) for every closed geometric point p of Y the fibre $X x_Y p$ is non-empty and simply connected.

Then X is simply connected.

THEOREM 2. Let X be a quasi-compact K-analytic space having an analytic reduction X which is irreducible and smooth over $\bar{K}$. Then X is simply connected.

3. EXAMPLES

(3.1) Every connected affinoid subset of $\mathbb{P}^1_K$ is simply connected. More generally, every connected analytic subspace of $\mathbb{P}^1_K$ is simply connected.

PROOF. The last statement follows from the first one and the observation that the connected analytic subspace X of $\mathbb{P}^1_K$ has an admissible covering $\{X_n | n \in \mathbb{N}\}$, with $X_n \subset X_{n+1}$ and each X_n is a connected affinoid subset of $\mathbb{P}^1_K$.

Let D be a connected affinoid subset of $\mathbb{P}^1_K$, let S be a locally constant sheaf of sets on D and let $\{D_1, \ldots, D_n\}$ be a covering of D by again connected affinoids such that $S|D_i$ is constant for each i. Now use [9] lemma (3.2) for D_{i_1}, D_{i_2} such that $D_{i_1} \cap D_{i_2} \neq \emptyset$. One finds that $S(D_j) \to S(D_{i_1} \cap D_{i_2})$ for $j = i_1$ and i_2 is bijective. Then also $S(D_{i_1} \cup D_{i_2}) \to S(D_j)$ is bijective for $j = i_1$ and i_2. Hence $S|D_{i_1} \cup D_{i_2}$ is constant. Repeated use of the lemma shows that S is a constant sheaf on D.

REMARK. This shows that the uniformization of a Mumford curve over any complete non-archimedean valued field K by an open connected subset of $\mathbb{P}^1_K$ is the universal covering.

(3.2) $\mathbb{B}^n = Spm(K\langle T_1, \ldots, T_n \rangle)$ is simply connected; generalized polydisks $D_1 \times \ldots \times D_m$ and monomially convex subsets of $\mathbb{B}^n$ are simply connected.

PROOF. Use (3.1) and Thm. 1 for suitable projections. The statement for $\mathbb{B}^n$ follows also from Thm. 2.

(3.3) Any extension G of an abelian variety X with good reduction with a split torus T is simply connected.

PROOF. X is simply connected by Thm. 2. Let $1 \to T \to G \xrightarrow{\pi} X \to 1$ denote the extension, where $T = \mathbb{G}_{m,K}^h$ for some h. The extension is given by a homomorphism of the charactergroup of T into $\mathrm{Pic}^0(X)(K)$. Since X has good reduction we can see X as an abelian scheme over K^0, the valuation ring of K. Further this implies that $\mathrm{Pic}^0(X)(K) = \mathrm{Pic}^0(X)(K^0)$. Let $\pi \in K^0$ satisfy $0 < |\pi| < 1$, then T_n is given by $\{(z_1, \ldots, z_h) \in \mathbb{G}_{m,K}^h \,|\, |\pi|^n \leq |z_i| \leq |\pi|^{-n}$ for all $i\}$. Let G_n denotes the extension of X by T_n. Using Thm. 1 and (3.2) one finds that G_n is simply connected. Further $G = \bigcup_1^\infty G_n$ is an admissible covering with:
(i) G_n symply connected.
(ii) $G_n \subset G_{n+1}$ for all n.
This proves that G is simply connected.

REMARK. An abelian variety Z over a field K, complete with respect to a discrete valuation admits (after a finite field extension) an analytic covering $G \to Z$ where G is an extension of an abelian variety with good reduction, by a split torus [2, 11]. This analytic covering is the universal one.

(3.4) Let k be a local field with uniformizing parameter π. Put $\Omega^{(2)} = \mathbb{P}^2$ – the set of k-rational lines of $\mathbb{P}^2$. Then $\Omega^{(2)}$ with the induced analytic structure is simply connected.

PROOF. In order to apply Thm. 1 we write $\Omega^{(2)} = \bigcup_{n=1}^\infty \Omega_n^{(2)}$ where $\Omega_n^{(2)}$ is defined as: $\{(x_0 : x_1 : x_2) |$for all $(a_0 : a_1 : a_2) \in \mathbb{P}^2(k)$ one has

$$|\textstyle\sum a_i x_i| \geq |\pi|^n \cdot \max(|a_0|, |a_1|, |a_2|) \max(|x_0|, |x_1|, |x_2|).\}$$

One easily sees that $\Omega_n^{(2)}$ is an affinoid subset of $\mathbb{P}^2$ and that $\Omega^{(2)} = \bigcup_1^\infty \Omega_n^{(2)}$ is an admissible covering.

We consider the morphism $\Omega_n^{(2)} \to \Omega_n^{(1)} \subset \mathbb{P}^1$ given by $(x_0 : x_1 : x_2) \longmapsto (x_0 : x_1)$ and $\Omega_n^{(1)} = \{(x_0 ; x_1) |$for all $(a_0 : a_1) \in \mathbb{P}^1(k)$ one has $|\sum_0^1 a_i x_i| \geq |\pi|^n \max(|a_0|, |a_1|) \cdot \max(|x_0|, |x_1|\}$.

Clearly $\Omega_n^{(1)}$ is a connected affinoid subset of $\mathbb{P}^1$ and the fibre for any closed geometric point p of $\Omega_n^{(1)}$ is a connected affinoid subspace of $\mathbb{P}_{K_p}^1$. Apply now Thm. 1.

(3.5) $\Omega^{(n)}$ denotes the open analytic subset of $\mathbb{P}^n$ given by $\Omega^{(n)} = \{(x_0 : x_1 : \ldots : x_n) | x_0, x_1, \ldots, x_n$ are linearly independent over $k\}$. $\Omega^{(n)}$ is simply connected.

PROOF. Use the projection $\Omega^{(n)} \to \Omega^{(n-1)}$ and proceed as in (3.4).

REMARK. D. Mumford [8] considers subgroups Γ of $PGl(3, k)$ having the property: Γ is discrete, co-compact and Γ has no elements $\neq 1$ of finite order. The group operates on $\Omega^{(2)}$ and the k-analytic space $\Gamma \backslash \Omega^{(2)}$ is isomorphic to an algebraic surface over k of general type. Again $\Omega^{(2)}$ is the universal covering of this surface.

(3.6) REMARK. For a one-dimensional connected regular analytic space X (say over an algebraically closed field) one can show the existence of a reduction $r: X \to \bar{X}$ such that every irreducible component of $\bar{X}$ is non-singular and every singular point of $\bar{X}$ is an ordinary double point. The fundamental group of X (in the analytic sense) coincides now with the fundamental groups of $\bar{X}$ with respect to the Zariski-topology.

This statement can be seen as an extension of Thm. 2 because one allows certain singular points in the reduction. This leads to the following interesting question:

Let X with $\dim X \geq 2$ and an analytic reduction $\bar{X}$ be given. What type of singularities can we allow $\bar{X}$ to have in order to conclude that X and $\bar{X}$ have the same fundamental group?

(3.7.) REMARK. The fundamental groups that one finds in the examples are very different:

(1) For Mumford-curves, the fundamental group is a free non-abelian group of finite rank.
(2) For Abelian varieties the fundamental group is a free finitely generated abelian group.
(3) For the Mumford examples of surfaces, the fundamental group is a group of finite presentation Γ such that Γ_{ab} is a finite group.

This leads to the question: Does there exist for every group Γ of finite presentation an algebraic variety with Γ as (K-analytic) fundamental group?

4. THE PROOFS

PROOF OF THM 1. Let $\{X_i | i = 1, \ldots, n\}$ be an admissible covering of X by connected affinoid subsets, let p be a closed geometric point of Y and let S be a locally constant sheaf on X such that $S|X_i$ is constant for each i.

Let $X_{i,j,t}$ $(t = 1, \ldots, s(i,j))$ denote the connected components of $X_i \cap X_j$. Put $A(i) = S(X_i)$ and $A(i,j,t) = S(X_{i,j,t})$. Then S is determined by the data: bijections $\lambda(i,j,t): A(i) \to A(i,j,t)$ (for all possible i, j, t where the sets are $\neq \emptyset$). The data should verify an obvious relation for triple intersections. The sheaf S is constant if and only if there exists a set A and bijections $\lambda(i): A \to A(i)$ making all diagrams (with nonempty $X_i \cap X_j$)

$$\begin{tikzcd} & A(i) \arrow[dr] & \\ A \arrow[ur, "\lambda(i)"] \arrow[dr, "\lambda(j)"'] & & A(i,j,t) \\ & A(j) \arrow[ur] & \end{tikzcd}$$ commutative.

The same data defines a locally constant sheaf on $X x_Y p = \cup X_i x_Y p$. According to our hypothesis this sheaf is constant.

This means that we have a solution $A \xrightarrow{\lambda(i)} A(i)$ of the problem above valid for all i, j, t with $X_{i,j,t} x_Y p \neq \emptyset$. For a suitable connected affinoid $T \in p$ one has $X_{i,j,t} x_Y p = \emptyset \Leftrightarrow X_{i,j,t} x_Y T = \emptyset$. Moreover T has a neighbourhood T' in Y such that again $X_{i,j,t} x_Y T = \emptyset \Leftrightarrow X_{i,j,t} x_Y T' = \emptyset$. It follows that the sheaf induced by S on $X x_Y T'$ is constant. If one chooses for every closed geometric point p some T_p in P and some neighbourhood T'_p of T_p in Y, then the compactness of the set of all geometric points of Y implies that Y is covered by finitely many of the T'_p's.

Hence the sheaf $\varphi_* S$ on Y is locally constant. By assumption the sheaf $\varphi_* S$ is constant. Since $\varphi_* S(Y) = S(X)$ it follows that S is constant.

PROOF OF THM 2. We will use the following algebraic result.

(4.1.) LEMMA. Let Z be a smooth and irreducible variety over a field k and let z be a point of Z. There exists an open affine neighbourhood U of z and a regular function $t: U \to \mathbb{A}^1$-{finitely many points}, such that every fibre of t is smooth and connected.

PROOF. We replace Z by some affine neighbourhood of z and embed this in a projective space $\mathbb{P}^n$ such that the Zariski-closure $\bar{Z}$ in $\mathbb{P}^n$ is normal. Since the singular locus of $\bar{Z}$ has codimension ≥ 2 we can apply Bertini's theorem to $\bar{Z}$ ([5], II 8.18). There exists a non-empty open subset O of hyperplanes in $\mathbb{P}^n$ intersecting $\bar{Z}$ into a non-singular and connected variety. ([5], III 7.9). Choose a hyperplane H_0 through z such that z is not singular on $H_0 \cap \bar{Z}$ and choose some $H_1 \in O$ with $H_1(z) \neq o$. Restriction of the rational map $\bar{Z} \to \mathbb{P}^1$, given by $u \longmapsto (H_0(u) : H_1(u))$, to suitable open subsets yields the required U and t.

It suffies to prove Thm. 2 locally on $\bar{X}$ for the Zariski-topology, because any locally constant sheaf on $\bar{X}$ is constant. Further Thm. 2 is known for X with $\dim X = 1$. We will assume $\dim X \geq 2$ and use induction with respect to $\dim X$. Using (4.1) we may assume that X is affinoid and that a regular function $\bar{t}: \bar{X} \to \bar{Y} = \mathbb{A}^1_{\bar{k}}$-{finitely many points} is given with smooth and connected fibres.

We lift $\bar{t}$ to a morphism of the affinoid algebra's:

$$K^0 \left\langle T, \frac{1}{Q(T)} \right\rangle \to O(X)^0 = K^0 \left\langle T, \frac{1}{Q(T)} \right\rangle \langle T_1, \ldots, T_m \rangle / (f_1, \ldots, f_s).$$

Here $Q(T)$ is a polynomial with Gauss-norm 1 such that

$$\bar{Y} = \operatorname{Spec}\left(\bar{K}\left[T, \frac{1}{\bar{Q}(T)}\right]\right).$$

The smoothness in the reduced case implies that the minors of the correct rank of $(\partial f / \partial T)$ generate the unit ideal in $O(X)^0$. For every closed geometric point p of

$$Y = Spm\left(K\left\langle T, \frac{1}{Q(T)}\right\rangle\right)$$

the fibre Xx_Yp is an affinoid space over K_p with a reduction which is still smooth over $\bar{K}_p$. Further Xx_Yp (and also the reduction of Xx_Yp) is connected. Otherwise Xx_YU would be disconnected for a suitable connected $U \in p$. For some point $v \in U$, then also Xx_Yv and its reduction is disconnected. However the reduction of Xx_Yv is a fibre of $\bar{t}$.

By induction the fibres Xx_Yp are simply connected. Apply now theorem 1.

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